Computing clique and chromatic number for circular-perfect graphs

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A celebrated result of Grötschel, Lovász and Schrijver [2] is that computing clique number $\omega(G)$ and chromatic number $\chi(G)$ of a perfect graph can be done in polynomial time.

This result relies on a polyhedral characterization of perfect graphs [1] showing that for such a graph G the stable set polytope $\mathrm{STAB}(G)$ coincides with its linear relaxation $\mathrm{QSTAB}(G)$. However, optimizing over $\mathrm{QSTAB}(G)$ does not work directly [2], but only via a detour involving a geometric representation of graphs. The resulting convex set $\mathrm{TH}(G)$ satisfies $\mathrm{STAB}(G) \subseteq \mathrm{TH}(G) \subseteq \mathrm{QSTAB}(G)$ and has the key property that $\vartheta(G) = \max \mathbb{1}^T x,\ x \in \mathrm{TH}(G)$ can be computed in polynomial time for any graph G [2]. For perfect graphs, $\mathrm{STAB}(G)$ and $\mathrm{TH}(G)$ coincide and, thus, the clique number equals $\omega(G) = \vartheta(\overline{G})$ which also allows to compute $\chi(G) = \omega(G)$ in polynomial time.

We address the question whether this result can be extended to graph classes $\mathcal{G}$ whose members G satisfy the best possible bound on the chromatic number for graph classes containing imperfect graphs, namely $\omega(G) \leq \chi(G) \leq \omega(G) + 1$. For such graphs $G \in \mathcal{G}$ clearly $\omega(G) \leq \vartheta(\overline{G}) \leq \chi(G) \leq \omega(G) + 1$ holds. Hence

$$\omega(G) = \lfloor \vartheta(\overline{G}) \rfloor \text{ and } \chi(G) = \lceil \vartheta(\overline{G}) \rceil$$

follows for $\vartheta(\overline{G}) \notin \mathbb{Z}$ and the two parameters can be computed in polytime for all graphs $G \in \mathcal{G}$, provided that we can decide which of the three cases

$$\begin{array}{l} \omega(G) < \lfloor \vartheta(\overline{G}) \rfloor = \chi(G) \\ \omega(G) = \lceil \vartheta(\overline{G}) \rceil < \chi(G) \\ \omega(G) = \lfloor \vartheta(\overline{G}) \rceil = \chi(G) \end{array}$$

occurs if $\vartheta(\overline{G}) \in \mathbb{Z}$ holds. We apply this method to a superclass of perfect graphs, the circular-perfect graphs defined by a more general coloring concept. The main result is that clique and chromatic number can be computed in polynomial time for all circular-perfect graphs [4] and, using similar techniques, further graph parameters for subclasses of circular-perfect graphs [3, 4].

In contrary, we exhibit that the same approach fails for two prominent graph classes $\mathcal{G}$ with the studied property, namely line graphs and planar graphs, unless $P = NP$.

References

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