Variations and extensions of transversals

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The concept of transversal of a family of subsets of a (finite) set has been extensively studied, under various denominations. We shall consider two possible extensions of this concept and exhibit their relations as well as some variations.

Given a finite ground set $V = \{v_1, \ldots, v_p\}$ and a property $\mathcal{P}$ defined on V, let $\mathcal{C}_V$ be the collection of all subsets C of V satisfying $\mathcal{P}$. Let $\nu(\mathcal{C}_V) = \max\{|C| C \in \mathcal{C}_V\}$ a subset C in $\mathcal{C}_V$ will be called **maximum** if $|C| = \nu(\mathcal{C}_V)$. Similarly $\lambda(\mathcal{C}_V) = \min\{|C| : C \in \mathcal{C}_V\}$ and C will be **minimum** if $|C| = \lambda(\mathcal{C}_V)$. A subset $C \in \mathcal{C}_V$ will be called **maximal** (resp. **minimal**) if it is inclusionwise maximal (resp. minimal) for property $\mathcal{P}$. A **transversal** is a subset $T \subseteq V$ with $|T \bigcap C| \geq 1$ for every maximum (or minimum, depending on the context) C in $\mathcal{C}_V$. In other words a transversal meets every maximum C (or every minimum C) in $\mathcal{C}_V$.

From here there are several ways of extending the concept of transversal. First given an integer d with $1 \leq d \leq \nu(\mathcal{C}_V)$ we say that a subset $T \subseteq V$ is a **d-transversal** if $|T \bigcap C| \geq d$ for each maximum C in C_v (or each minimum C in $\mathcal{C}_v$ with $d \leq \lambda(\mathcal{C}_V)$). With this definition a transversal is a 1-transversal.

If T is a transversal, it meets every maximum C in $\mathcal{C}_V$; so if we remove the elements of T from V, then the maximum size $\nu(\mathcal{C}_{V-T})$ of subsets C of $V - T$ having property $\mathcal{P}$ will satisfy $\nu(\mathcal{C}_{V-T}) \leq \nu(\mathcal{C}) - 1$. This suggests another way of extending the notion of transversal: given an integer d (with $1 \leq d \leq \nu(\mathcal{C})$ a subset $B \subseteq V$ is a **d-blocker** if $|C| \leq \nu(\mathcal{C}_V) - d$ for every subset C of $V - B$ having property $\mathcal{P}$. So we have $\nu(\mathcal{C}_{V-B}) \leq \nu(\mathcal{C}_V) - d$.. When we consider subsets C of minimum size, a **d-blocker** B is such that $\lambda(\mathcal{C}_{V-B}) \geq \lambda(\mathcal{C}_V) - d$. Again we notice that for $d = 1$, a 1-blocker is a transversal. In fact the concepts of d-blockers and d-transversals are equivalent for $d = 1$. But they may be different for $d \geq 2$.

There are many examples of d-transversals and d-blockers which have been examined:

a) V can be the set of arcs of a directed graph with a source s and a sink t. $\mathcal{C}_V$ is the family of all $s - t$ paths and a subset C of V verifies property $\mathcal{P}$ if it forms an $s - t$ path. A d-transversal is a subset $T \subseteq V$ such that for every shortest $s - t$ path C in G we have $|C \bigcap T| \geq d$. It follows from known results by Wagner and by Khachiyan et al., that a minimum d-transversal of all shortest $s - t$ paths can be found in polynomial time. If we denote by $\ell^*(G)$ the length of a shortest $s - t$ path, (i.e., the number of arcs in the path) then a d-blocker B is a subset of arcs such that in G' constructed on $V - B$ every $s - t$ path has length at least $\ell^*(G) + d$. It has been shown by Khachiyan et al. that finding a minimum d-blocker is NP-hard.

b) If V is the ground set of a **matroid** M and $\mathcal{C}_V$ the collection of independent sets, the maximum independent sets are the bases of M. Their cardinality is $r(M)$ where r is the **rank** function of M. A d-transversal is a subset $T \subseteq V$ such that for any basis C of

M we have $|T \bigcap C| \geq d$. A d-blocker is a subset $B \subseteq V$ such that any basis $C \subseteq V - B$ has $|C| \leq r(M) - d$. It can be shown that for any $d \geq 1$ a subset $T \subseteq V$ in a matroid on V is a d-transversal if and only if it is a d-blocker.

c) One may also consider $s - t$ cuts in a directed graph with a source s and a sink t. An $s - t$ cut is defined by a subset A of vertices of G such that $s \in A, t \notin A$; it consists of all arcs (u, v) with $u \in A, v \notin A$. V will be the set of arcs of G and $\mathcal{C}_V$ the collection of all $s - t$ cuts in G. A d-transversal is a subset T of arcs such that every minimum $s - t$ cut C has $|C \bigcap T| \geq d$. Notice however that removing a subset B of arcs may not increase the minimum cardinality of an $s - t$ cut, so d-blockers cannot be defined in the usual way. One may however consider the problem of adding (instead of removing) a minimum subset of arcs in order to increase by at least d the minimum capacity of an $s - t$ cut. Network flow techniques can be used for this.

d) Let now G be a graph constructed on a set V of edges; $\mathcal{C}_V$ is the collection of all **matchings** in G. A d-transversal is a subset T of the edge set T such that each maximum matching C of G satisfies $|C \bigcap T| \geq d$ and a d-blocker is a subset B of edges such that in the subgraph $G^{'}$ obtained by removing the edges of B we have $\nu(\mathcal{C}_{V-B}) \leq \nu(\mathcal{C}_V) - d$, i.e. we have reduced the maximum size of a matching in $G^{'}$ by at least d. It has been shown that finding minimum d-transversals and minimum d-blockers in NP-hard even if $d = 1$ and G is bipartite. In special cases (chains, cycles, complete graphs or complete bipartite graphs), one can construct such subsets in polynomial time. In the case of complete graphs (bipartite or not) the concepts of d-transversals and d-blockers are identical. For grid graphs, there are constructions giving minimum d-transversals and d-blockers, and for trees a dynamic programming procedure does provide a construction of such subsets.

e) Finally an interesting case is when V is the vertex set of a graph, a subset C satisfies property $\mathcal{P}$ if it is a stable set. A d-transversal is a subset T of vertices such that for every maximum stable set C we have $|C \bigcap T| \geq d$. Then $\nu(\mathcal{C}_V)$ will be the stability number of G (usually denoted by $\lambda(G)$) and a d-blocker B is a subset of vertices in G whose removal decreases by at least d the maximum size of a stable set. It has been shown that for split graphs, there are only two possible values for the minimum size of a d-blocker but it is NP-complete to decide between the two values. For constructing minimum d-transversals in split graphs, there is a polynomial algorithm due to C.M. Lee. This holds also for complements of balanced, of strongly chordal, of triangle free graphs (for fixed $d \geq 2$). For G bipartite, it is possible to characterize the sets of vertices which are in all (resp. in none of the) maximum stable sets. Using these characterizations, one can construct in polynomial time minimum d-transversals and d-blockers.

There are many open questions which remain in this area and we intend to present some. In particular it would be interesting to examine the case of graphs generalizing bipartite graphs where maximum stable sets can be found in polynomial time. The weighted case of all these problems will be considered and some complexity results will be stated for a few cases (split graphs, cobipartite graphs).

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