Forbidden induced subgraph characterizations of graph classes

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In this talk we present partial characterizations of different graph classes by forbidden induced subgraphs. First, we analyze two known classes of intersection graphs: *circular-arc graphs* and *circle graphs.* Circular-arc graphs (CA), defined by Klee in 1969, are the intersection graphs of arcs on a circle. Proper circular-arc graphs, an important subclass of circular-arc graphs, were characterized by minimal forbidden induced subgraphs by Tucker in 1974. But the problem of characterizing the whole class of CA graphs by forbidden induced subgraphs remains open. We will show some steps in this direction. We obtained minimal forbidden induced subgraphs characterizations for CA graphs when the graph belongs to different subclasses: P_4-free graphs, paw-free graphs, claw-free chordal graphs and diamond-free graphs. We are interested now in the characterization of this class when the graph is $3K_1$-free or claw-free.

Circle graphs are the intersection graphs of chords on a circle. They were introduced by Even and Itai in 1971, where an application to solve a problem proposed by Knuth is shown. In 1994 Bouchet characterized this class using forbidden induced subgraphs and local equivalence. Recently, Geelen and Oum showed another characterization in terms of pivoting. But none of these results solve the problem of characterizing the whole class of circle graphs by forbidden induced subgraphs. As we did for circular-arc graphs, we obtained new characterizations of this class by minimal forbidden induced subgraphs for linear domino graphs, P_4-tidy graphs, and tree-cographs. We also characterized by minimal forbidden induced subgraphs the class of unit Helly circle graphs, which are those circle graphs having a model whose chords have all the same length, are pairwise different, and satisfy the Helly property.

Furthermore, we study three subclasses of graphs related to perfect graphs. A graph is *perfect* if the chromatic number χ is equal to the clique number ω for every induced subgraph or, equivalently, if the independence number α is equal to the clique-covering number k for every induced subgraph. A graph G is *K-perfect* if its clique graph $K(G)$ is perfect. The classes analyzed are *coordinated* and *balanced graphs* (both subclasses of perfect graphs); and clique-perfect graphs, a variation of perfect graphs.

A *clique-transversal* of a graph G is a subset of vertices that meets all the cliques of G. A *clique-independent set* is a collection of pairwise vertex-disjoint cliques. The *clique-transversal number* and *clique-independence number* of G, denoted by $\tau_C(G)$ and $\alpha_C(G)$, are the sizes of a minimum clique-transversal and a maximum clique-independent set of G, respectively. Clearly, $\alpha_C(G) \leq \tau_C(G)$, for any graph G. A graph G is *clique-perfect* if $\tau_C(H) = \alpha_C(H)$ for every induced subgraph H of G. It is easy to see that $\alpha_C(G) = \alpha(K(G))$ and $\tau_C(G) \geq k(K(G))$ (and the equality holds if G is clique-Helly). So, there is a connection between clique-perfect and K-perfect graphs. Nevertheless. they are not the same class, even restricted to clique-Helly graphs. Clique-perfect graphs have been implicitly studied in several works but the terminology "clique-perfect" has been introduced by Guruswami and

Pandu-Rangan in 2000.

In an analogous way, we define a *K-coloring* of a graph as an assignment of colors to its cliques in such a way that no two cliques with non-empty intersection receive the same color. A *Helly K-complete* of a graph G is a collection of cliques of G with common intersection. The *K-chromatic number* and *Helly K-clique number* of G, denoted by $F(G)$ and $M(G)$, are the sizes of a minimum K-coloring and a maximum Helly K-complete of G, respectively. Clearly, $F(G) = \chi(K(G))$ and $M(G) \leq \omega(K(G))$ (and the equality holds if G is clique-Helly). It is easy to see also that $F(G) \geq M(G)$ for any graph G. A graph G is *C-good* if $F(G) = M(G)$. A graph G is *coordinated* if every induced subgraph of G is C-good. Again, there is a connection between coordinated and K-perfect graphs but, also in this case, they are not the same class, even restricted to clique-Helly graphs. We proved instead that coordinated graphs are perfect.

The complete lists of minimal forbidden induced subgraphs for the classes of clique-perfect and coordinated graphs are also unknown, but some partial characterizations have been obtained in different works. In 1986, Lehel and Tuza characterized clique-perfect graphs by forbidden induced subgraphs when the graph is chordal. We obtained the complete list of minimal forbidden induced subgraphs for clique-perfect graphs when the graph belongs to the following classes: diamond-free graphs, paw-free graphs, P_4-tidy graphs, hereditary clique-Helly claw-free graphs, Helly circular-arc graphs, line graphs, and complements of line graphs. For coordinated graphs we did this task for these classes: paw-free graphs, {gem,W_4,bull}-free graphs, line graphs, and complements of forests.

Another characterization of perfect graphs, due to Chvátal, is the following. A graph is *perfect* if and only if its clique-matrix is perfect. In analogy to perfect graphs, *balanced graphs* were defined by Dalhaus, Manuel and Miller in 1998 as those graphs whose clique-matrix is balanced. Since balanced matrices are also perfect, balanced graphs form a subclass of perfect graphs. It also can be proved that they are clique-perfect, coordinated and K-perfect. We obtained a complete characterization of balanced graphs by means of forbidden induced subgraphs. However, this characterization is not by *minimal* forbidden induced subgraphs. So, we addressed the problem of finding the *minimal forbidden induced subgraphs* for this class. We did this task when the graph belongs to these classes: paw-free graphs, P_4-tidy graphs, line graphs, complements of line graphs, Helly circular-arc graphs, gem-free circular-arc graphs, and claw-free circular-arc graphs.

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