Tight lower bounds for the size of epsilon-nets

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According to a well known theorem of Haussler and Welzl (1987), any range space of bounded VC-dimension admits an ε-net of size $O\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$. Using probabilistic techniques, Pach and Woeginger (1990) showed that there exist range spaces of VC-dimension 2, for which the above bound can be attained. The only known range spaces of small VC-dimension, in which the ranges are geometric objects in some Euclidean space and the size of the smallest ε-nets is superlinear in $\frac{1}{\varepsilon}$, were found by Alon (2010). In his examples, the size of the smallest ε-nets is $\Omega\left(\frac{1}{\varepsilon}g(\frac{1}{\varepsilon})\right)$, where g is an extremely slowly growing function, closely related to the inverse Ackermann function.

We show that there exist geometrically defined range spaces, already of VC-dimension 2, in which the size of the smallest ε-nets is $\Omega\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$. We also construct range spaces induced by axis-parallel rectangles in the plane, in which the size of the smallest ε-nets is $\Omega\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$. By a theorem of Aronov, Ezra, and Sharir (2010), this bound is tight.

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